Logic
Resolution in First-order logic

Few laws related to propositional knowledge are stated below:

(i) Idempotency : $P \vee P = P$

: $P \wedge P = P$

(ii) Commutative law : $P \vee Q = Q \vee P$

: $P \wedge Q = Q \wedge P$

: $P \leftrightarrow Q = Q \leftrightarrow P$

(iii) Associative law : $(P \vee Q) \vee R = P \vee (Q \vee R)$

: $(P \wedge Q) \wedge R = P \wedge (Q \wedge R)$

(iv) Distributive law : $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

: $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

(v) De Morgan's rule : $\sim (P \vee Q) = \sim P \wedge \sim Q$

: $\sim (P \wedge Q) = \sim P \vee \sim Q$

(vi) Implication removal : $P \leftrightarrow Q = \sim P \vee Q$

(vii) Biconditional elimination : $P \rightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P)$

(viii) Absorption law : $P \vee (P \wedge Q) \equiv P$, $P \wedge (P \vee Q) \equiv P$

(ix) Contrapositive : $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$

(x) Double negation : $P \equiv \neg (\neg P)$

(xi) Fundamental identities :

(a) $P \vee \neg P \equiv T$

(b) $P \wedge \neg P \equiv F$

(c) $P \vee T \equiv T$

(d) ~~$P \vee T \equiv P$~~ $P \wedge T \equiv P$

(e) $P \vee F \equiv P$

(f) ~~$P \vee F \equiv F$~~ $P \wedge F \equiv P$

(g) $(P \Rightarrow Q) \wedge (P \Rightarrow \neg Q) \equiv \neg P$

(h) $P \Rightarrow Q \equiv (\neg P \vee Q)$

Properties of Statements

- Valid
- Satisfiable
- Unsatisfiable
- Equivalence
- Logical Consequence

2.4 Properties of Statements

Before moving ahead, let us discusses some properties of propositional calculus statements or WFFs described as follows:

(i) ***Valid***: A sentence is valid, if it is true for all values of inputs or for every interpretation. An all true statement is also called tautology. For example, $P \vee \neg P$ is valid since every interpretation of P results in a true value for $P \vee \neg P$.

(ii) ***Satisfiable***: A statement having at least one interpretation for which it is true, is called Satisfiable. For example, if statement P is Satisfiable, it will have at least one interpretation of P for which the value of P is true. However, P will not necessarily be valid because it is not true for every interpretation of P i.e., a value F for P will result in a value F for sentence P.

(iii) ***Unsatisfiable*** (or ***contradiction***): A statement or preposition is called Unsatisfiable if there is no interpretation for which it is true. For example, $P \wedge \neg P$ is unsatisfiable because it is false for every interpretation of P.

(iv) ***Equivalence***: Two statements s_1 and s_2 are equivalent if for every interpretation they have the same truth-value. For example, two statements P and $\neg(\neg P)$ are equivalent since both have the same truth-value for every interpretation of P.

(v) ***Logical consequence***: Statement s_2 is said to be logical consequence of s_1, if it is satisfied by all interpretations which satisfy s_1. For example, out of given two sentences P and $P \wedge Q$, P is said to be logical consequence of $P \wedge Q$ because for every interpretation for which $P \wedge Q$ is true, P is also true.

Inference in Propositional Logic

- **Addition**: From a given statement P, infer $P \vee Q$, where Q can be any other statement. This is also written as:

$$\frac{P}{\therefore (P \vee Q)}$$

For example,

Given : Adwet is an obedient boy

Conclude : Adwet is an obedient boy or Sushant is a lazy boy

This rule can be represented in implication form as $P \rightarrow (P \vee Q)$.

- **Conjunction**: From given two sentences or statements P and Q, infer $P \wedge Q$, or:

$$\frac{\begin{array}{c} P \\ Q \end{array}}{\therefore (P \wedge Q)}$$

For example,

Given : Vishal is an intelligent student

And : Shyam is a good player

Conclude : Vishal is an intelligent student and Shyam is a good player

Implication form of this rule is represented as $P \wedge Q \rightarrow (P \wedge Q)$.

- **Simplification**: From given sentence $P \wedge Q$, infer P, or:

$$\frac{P \wedge Q}{\therefore P}$$

For example,

Given : Kate is a beautiful woman and John is an ugly man
Conclude : Kate is a beautiful woman

This rule can be represented in implication form as $(P \wedge Q) \rightarrow P$.

- **Modus Ponens**: From given two statements P and $P \rightarrow Q$, infer Q. This is also written as:

$$\begin{array}{c} P \\ \underline{P \rightarrow Q} \\ \therefore Q \end{array}$$

For example:

given : Adwet is intelligent
and : Adwet is intelligent → Adwet tops the class
conclude : Adwet tops the class

This rule is written in implication form as $(P \wedge (P \rightarrow Q)) \rightarrow Q$.

- **Modus tollens:** From the two given statements $\neg Q$ and $(P \rightarrow Q)$, infer $\neg P$, or:

$$\begin{array}{c} \neg Q \\ P \rightarrow Q \\ \hline \therefore \neg P \end{array}$$

For example,

Given : Justin is not a religious person
And : Justin goes to church daily implies Justin is a religious person
Conclude : Justin does not go to church daily

Implication form of this rule is represented as $(\neg Q \wedge (P \rightarrow Q)) \rightarrow \neg P$.

- **Chain rule or Hypothetical Syllogism**: From $(P \rightarrow Q)$ and $(Q \rightarrow R)$, infer $(P \rightarrow R)$, or

$$\begin{array}{c} P \rightarrow Q \\ Q \rightarrow R \\ \hline \therefore (P \rightarrow R) \end{array}$$

For example,

Given : India has natural resources → India can generate energy
And : India can generate energy → India is prosperous country
Conclude : India is prosperous country

This rule is represented in implication form as $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$.

- **Disjunctive syllogism**: From two given sentences $\neg P$ and $(P \vee Q)$, infer Q, or:

$$\begin{array}{c} \neg P \\ P \vee Q \\ \hline \therefore Q \end{array}$$

For example,

Given : Mohit is not a laborious boy
And : Mohit is a laborious boy or Suchi is an honest girl
Conclude : Suchi is an honest girl

Implication form of this rule is written as $(\neg P \wedge (P \vee Q)) \rightarrow Q$.

- **Constructive dilemma**: From given two sentences $((P \rightarrow Q) \wedge (R \rightarrow S))$ and $(P \vee R)$, infer $(Q \vee S)$, or:

$$\begin{array}{c} (P \rightarrow Q) \wedge (R \rightarrow S) \\ P \vee R \\ \hline \therefore (Q \vee S) \end{array}$$

For example,

Given : (Bret loves Kate implies Kate loves Bret) and (Jash hates Sushi implies Sushi hates Jash)
And : Bret loves Kate or Jash hates Sushi
Conclude : Kate loves Bret or Sushi hates Jash

This rule is represented in implication form as $(((P \rightarrow Q) \wedge (R \rightarrow S)) \wedge (P \vee R)) \rightarrow (Q \vee S)$.

- **Destructive dilemma**: From given two sentences $((P \rightarrow Q) \wedge (R \rightarrow S))$ and $(\neg Q \vee \neg S)$, infer $(P \vee R)$, or:

$$\begin{array}{c} (P \rightarrow Q) \wedge (R \rightarrow S) \\ \neg Q \vee \neg S \\ \hline \therefore (P \vee R) \end{array}$$

For example,

Given : Albart scored 85% marks implies Albart is an intelligent student and Steffi scored 54% marks implies Steffi is a weak student
and : Albart is not an intelligent student or Steffi is not a weak student
conclude : Albart scored 85% marks or Steffi scored 54% marks

Assignment

1	Find the truth value of following propositions: (i) If 2 is not an integer, then ½ is an integer. (ii) If 2 is an integer, then ½ is an integer.
2	Translate the following sentences into propositional forms: (a) If it is not raining and I have time, then I will go to a movie. (b) If it is raining and I will not go to a movie. (c) It is not raining. (d) I will not go to a movie. (e) I will not go to a movie only if it is not raining.
3	If P, Q, R are the propositions, defined as above. Write the sentences in English corresponding to the following propositional forms: (i) $(\neg P \wedge Q) \leftrightarrow R$ (ii) $(Q \rightarrow R) \wedge (R \rightarrow Q)$ (iii) $\neg (Q \vee R)$ (iv) $R \rightarrow \neg P \wedge Q$

Without using truth tables, prove that
$\neg(p \rightarrow q) \rightarrow \neg q$ is a tautology.

Resolution

Truth Table				
P	Q	Conjunction AND P Λ Q	Disjunction OR P v Q	Negation NOT P ~P
T	T	T	T	F
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

- Variable: A variable is simply a letter that can be either true or false.
- Literal: A literal is either a variable or the negation of a variable.
- Sum and Product: A disjunction of literals is called a sum and a conjunction of literals is called a product.
- Clause: A clause is a disjunction of literals.

Clauses are usually written as follows, where the symbols l_i are literals:

$$l_1 \vee \cdots \vee l_n$$

Horn Clause

- A Horn clause is a clause (a disjunction of literals) with at most one positive literal. $\neg p \lor \neg q \lor ... \lor \neg t \lor u$
- Conversely, a disjunction of literals with at most one negated literal is called a **dual-Horn clause.**
- A Horn clause with exactly one positive literal is a **definite clause** or a **strict Horn clause.**
- a definite clause with no negative literals is a **unit clause**
- a unit clause without variables is a **fact**;
- A Horn clause without a positive literal is a **goal clause.**
- Note that the empty clause, consisting of no literals (which is equivalent to false) is a goal clause.

Resolution

Disjunctive Normal Forms (DNF):

A formula which is equivalent to a given formula and which consists of a **<u>sum</u>** of elementary products is called a disjunctive normal form of given formula.

Example : $(P \land \sim Q) \lor (Q \land R) \lor (\sim P \land Q \land \sim R)$

Conjunctive Normal Form (CNF):

A formula which is equivalent to a given formula and which consists of a **product** of elementary products is called a conjunctive normal form of given formula.

Example : (P~ ∨ Q) ∧ (Q ∨ R) ∧ (~ P ∨ Q ∨ ~ R)

If every elementary sum in CNF is tautology, then given formula is also tautology.

Principle Disjunctive Normal Form (PDNF) :

An equivalent formula consisting of **disjunctions of minterms** only is called the principle disjunctive normal form of the formula.

It is also known as **sum-of-products** canonical form.

Example:

$(P \land \sim Q \land \sim R) \lor (P \land \sim Q \land R) \lor (\sim P \land \sim Q \land \sim R)$

- The minterm consists of conjunctions in which each statement variable or its negation, but not both, appears only once.
- The minterms are written down by including the variable if its truth value is T and its negation if its truth value is F.

Principle Conjunctive Normal Form (PCNF) :

An equivalent formula consisting of **conjunctions of maxterms** only is called the principle conjunctive normal form of the formula.
It is also known as product-of-sums canonical form.

Example :
$(P \lor \sim Q \lor \sim R) \land (P \lor \sim Q \lor R) \land (\sim P \lor \sim Q \lor \sim R)$

- The maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once.
- The dual of a minterm is called a maxterm.
- Each of the maxterm has the truth value F for exactly one combination of the truth values of the variables.
- The maxterms are written down by including the variable if its truth value is F and its negation if its truth value is T.

Resolution Proof Example.

(a) Marcus was a man.

(b) Marcus was a Roman.

(c) All men are people.

(d) Caesar was a ruler.

(e) All Romans were either loyal to Caesar or hated him (or both).

(f) Everyone is loyal to someone.

(g) People only try to assassinate rulers they are not loyal to.

(h) Marcus tried to assassinate Caesar.

Steps to Convert to CNF (Conjunctive Normal Form)

CNF

In Boolean logic, a formula is in conjunctive normal form (CNF) or clausal normal form if it is a conjunction of one or more clauses, where a clause is a disjunction of literals; otherwise put, it is a product of sums or an **AND of ORs**. As a canonical normal form, it is useful in automated theorem proving and circuit theory.

A sentence expressed as a **conjunction of disjunctions of literals** is said to be in **Conjunctive normal Form** or CNF.

Examples and non-examples

All of the following formulas in the variables A, B, C, D, E, and F are in conjunctive normal form:

- $(A \lor \neg B \lor \neg C) \land (\neg D \lor E \lor F)$
- $(A \lor B) \land (C)$

The following formulas are **not** in conjunctive normal form:

- $\neg(B \lor C)$, since an OR is nested within a NOT
- $(A \land B) \lor C$

Steps to Convert to CNF

Conversion FOL to CNF

Step 1: Eliminate Biconditionals and Implications:

- Eliminate $\rightarrow$, replacing $P \rightarrow Q$ with $(\neg P \vee Q)$
- Eliminate $\leftrightarrow$, replacing $P \leftrightarrow Q$ with
  $(P \rightarrow Q) \wedge (Q \rightarrow P)$
  $\equiv [\neg P \vee Q] \wedge [\neg Q \vee P]$

Step 2: Move all Negations $(\neg)$ inwards.

- $\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$
- $\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg\neg P(x) \equiv P(x)$

Step 3: Standardize Variables apart by renaming them: each quantifier should use a different variable.

For sentence like $(\forall x\, P(x)) \vee (\exists x\, Q(x))$ which use the same variable name twice, change the name of one of the variables.

Steps to Convert to CNF

Step4: Skolemize: Each existential variables is replaced by a Skolem Constant or skolem function of enclosing Universally quantified variables:

* For instane, $\exists x\, Rich(x)$ becomes $Rich(G1)$
  Where G1 is a new Skolem constant

* "Everyone has a heart"

$$\forall x\; Person(x) \rightarrow \exists y\; Heart(y) \wedge Has(x,y)$$

becomes,

$$\forall x\; Person(x) \rightarrow Heart(H(x)) \wedge Has(x, H(x))$$

where H is a new symbol (skolem function)

Step 5: Drop universal Quantifiers.

* For instance, $\forall x\; Person(x)$ becomes $Person(x)$

Step 6: Distribute $\wedge$ over $\vee$:

* $(P \wedge Q) \vee S \equiv (P \vee S) \wedge (Q \vee S)$

Example 1:

Assume the following facts:

i. Steve likes ALL easy courses.

ii. Science courses are hard.

iii. All the courses in the basketweaving department are easy.

iv. BK301 is a basketweaving course.

Use resolution to answer the question:

"What course would Steve like?"

Example 1: Solution

Assume the following facts:

i. Steve likes ALL easy courses.

ii. Science courses are hard.

iii. All the courses in the basketweaving department are easy.

iv. BK301 is a basketWeaving course.

Use resolution to answer the question:

"What course would steve like?"

Convert into FOL:

i. $\forall x\ easy(x) \rightarrow likes(steve, x)$

ii. $\forall x\ science(x) \rightarrow \neg easy(x)$

iii. $\forall x\ basketWeaving(x) \rightarrow easy(x)$

iv. $basketWeaving(BK301)$

The conclusion is encoded as:

$likes(steve, x)$

Convert into CNF:

i. $\neg easy(x) \lor likes(steve, x)$

ii. $\neg science(x) \lor \neg easy(x)$

iii. $\neg basketWeaving(x) \lor easy(x)$

iv. $basketWeaving(BK301)$

Example 1: Solution

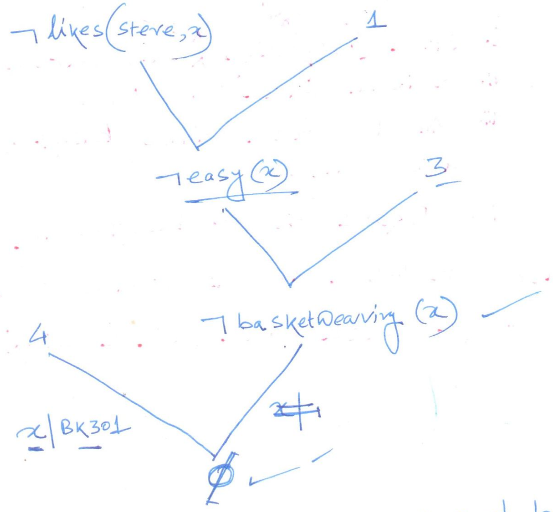
Convert into CNF:

i. ¬easy(x) ∨ likes(steve, x)

ii. ¬science(x) ∨ ¬easy(x)

iii. ¬basketWeaving(x) ∨ easy(x)

iv. basketWeaving(BK301)



The substitution x|BK301 is produced by the unification algorithm which says that only wff of the form like(steve, x) which follows from the premises is

likes(steve, BK301)

Convert to First order Logic

(a) Marcus was a man.

(b) Marcus was a Roman.

(c) All men are people.

(d) Caesar was a ruler.

(e) All Romans were either loyal to Caesar or hated him (or both).

(f) Everyone is loyal to someone.

(g) People only try to assassinate rulers they are not loyal to.

(h) Marcus tried to assassinate Caesar.

(a) man(marcus)

(b) roman(marcus)

(c) $\forall X.\ man(X) \rightarrow person(X)$

(d) ruler(caesar)

(e) $\forall X.\ roman(x) \rightarrow loyal(X,caesar) \vee hate(X,caesar)$

(f) $\forall X \exists Y.\ loyal(X,Y)$

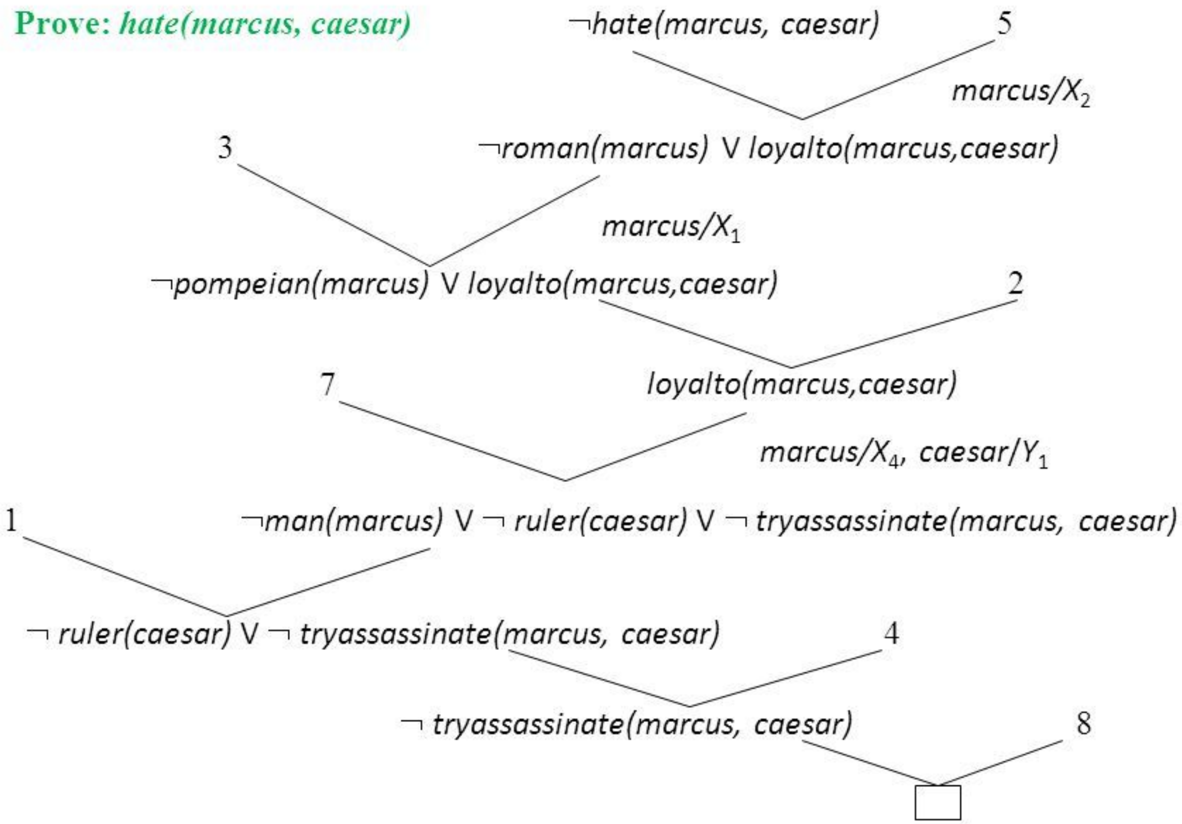
(g) $\forall X \forall Y.\ person(X) \wedge ruler(Y)\ tryassasin(X,Y) \rightarrow \neg loyal(X,Y)$

(h) tryassasin(marcus,caesar)

Convert to Clausal Form

1. man(marcus)
2. roman(marcus)
3. (¬man(X), person(X))
4. ruler(caesar)
5. (¬roman(X), loyal(X,caesar), hate(X,caesar))
6. (loyal(X,f(X))
7. (¬person(X), ¬ruler(Y), ¬tryassasin(X,Y), ¬loyal(X,Y))
8. tryassasin(marcus,caesar)

Resolution Proof



Types of Resolution

Resolution Strategies:

Unit Resolution

* Every resolution step must involve a unit Clause.
* Leads to a good Speed up.
* Incomplete in general.
  - [use the unit clause in every step may not get the result/ proof the result.]

but

* Complete for Horn knowledge bases.

Input Resolution:

* Every resolution step must involve a input sentence (from the query or the knowledge Base)
  → Can not resolve from two derived clauses.

* Always start with goal clause, assuming that the KnowledgeBase itself consistent, addition of the goal is expected to make it incosistent.

* So we start from the goal and keep on doing this resoultion at each step at least one of the clauses should be original KB or goal.

* In Horn ~~clause~~ knowledge bases, Modus Ponens is a kind of input resolution strategy. (P and ¬P ∨ Q resolvant Q)
* Incomplete in general. (if you try to solve from two derived clauses)
* complete for Horn Knowledge bases.

* Generalization of input resolution which is complete is called Linear Resolution.

Linear Resolution:

* slight generalization of input resolution
* Allow P and Q to be resolved together either if P is in the original KB or if P is an ancestor of Q in the proof tree.
* Linear Resolution is complete.

Thank You!

Any Questions?